EXERCISE SHEET

1) Using one of the conjectures in Lecture 1, show that $H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(n))$ is supposed to agree with $H^i_{\mathcal{M}}(X, \mathbb{Q}(n))$ for $n > i$.

2) In the lecture I never defined homological equivalence. Here is the definition: Let X be a smooth projective variety over a field $k \subseteq \mathbb{C}$ (for simplicity, and this is all we need. Write $Z^n(X) := Z^n(X, 0)$ for the codimension n cycles on X. We say that two cycles $Z, Z' \in Z^n(X)$ are *homologically equivalent*, and write $Z \sim_{\mathrm{hom}} Z'$, if $\mathrm{cl}_{\mathrm{B}}(Z) = \mathrm{cl}_{\mathrm{B}}(Z')$ where
$$\mathrm{cl}_{\mathrm{B}} : Z^n(X) \to H^{2n}_{\mathrm{sing}}(X_{\mathbb{C}}, \mathbb{Z}(n))$$
is the Betti cycle class map. For the sake of asking a question: Give an example of two cycles Z, Z' such that $Z \sim_{\mathrm{hom}} Z'$ but $Z \not\sim_{\mathrm{rat}} Z'$.

3) Let X be a smooth projective variety over $\mathbb{Q}$, and let $n \geq 0$. Recall that the Tate conjecture in this setting asserts first that
$$\mathrm{rank}_{\mathbb{Z}}\, N^n(X) = \dim_{\mathbb{Q}_\ell} H^{2n}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$
where ℓ is any prime, and second that the right hand side is
$$\dim_{\mathbb{Q}_\ell} H^{2n}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(n))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = -\mathrm{ord}_{s=n+1} L(H^{2n}(X), s)\,.$$
Show that Beilinson's conjectures together with the Tate conjecture predicts that
$$\dim_{\mathbb{Q}} H^{2n+1}_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(n)) = \mathrm{ord}_{s=n} L(H^{2n}(X), s)\,.$$

4) Let E be an elliptic curve over $\mathbb{Q}$. Show that $H^3_{\mathcal{M}/\mathbb{Z}}(E, \mathbb{Q}(2))$ is supposed to be trivial.